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NPSOR-91-16

NAVAL POSTGRADUATE SCHOOL

Monterey, California

A CRITIQUE OF DISTRIBUTIONAL ANALYSIS IN SOCIAL CHOICE

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May 1991

Prepared for:
National Research Council and National Science Foundation

NAVAL POSTGRADUATE SCHOOL,
MONTEREY, CALIFORNIA

Rear Admiral R. W. West, Jr.
Superintendent

Harrison Shull
Provost

This report was prepared in conjunction with research funded by the National Research Council and National Science Foundation.

This report was prepared by:

Unclassified

Security Classification of this page

REPORT DOCUMENTATION PAGE

1a Report Security Classification UNCLASSIFIED	1b Restrictive Markings
2a Security Classification Authority	3 Distribution Availability of Report Approved for public release; distribution is unlimited
2b Declassification/Downgrading Schedule	
4 Performing Organization Report Number(s) NPSOR-91-16	5 Monitoring Organization Report Number(s)
6a Name of Performing Organization Naval Postgraduate School 6b Office Symbol *(If Applicable)* OR	7a Name of Monitoring Organization
6c Address *(city, state, and ZIP code)* Monterey, CA 93943-5000	7b Address *(city, state, and ZIP code)*
8a Name of Funding/Sponsoring Organization National Research Council and NSF 8b Office Symbol *(If Applicable)* OR/Tv	9 Procurement Instrument Identification Number OM&N DIRECT FUNDING
8c Address *(city, state, and ZIP code)* Washington, DC 20418	10 Source of Funding Numbers: Program Element Number / Project No / Task No / Work Unit Accession No

11 Title *(Include Security Classification)* A CRITIQUE OF DISTRIBUTIONAL ANALYSIS IN SOCIAL CHOICE

12 Personal Author(s) Craig A. Tovey

13a Type of Report	13b Time Covered	14 Date of Report *(year, month,day)*	15 Page Count
Technical	From To	1991, May	

16 Supplementary Notation The views expressed in this paper are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.

17 Cosati Codes			18 Subject Terms *(continue on reverse if necessary and identify by block number)*
Field	Group	Subgroup	Spatial model; social choice; asymptotics; consistency; probability; voting

19 Abstract *(continue on reverse if necessary and identify by block number*

Distributional analysis is widely used to study social choice in Euclidean models [28, 29, 1, 3, 8, 15, 5, 2, e.g.]. This method assumes a continuum of voters distributed according to a distribution function. Since infinite populations do not exist, the goal of distributional analysis is to give insight into the behavior of large finite populations. However, properties of finite populations do not in general converge to the properties of infinite populations. Thus the method of distributional analysis is flawed. In some cases it will predict that a point is in the core with probability 1, while the true probability converges to 0. On the other hand, it is sometime possible to combine distributional analysis with probabilistic analysis to make correct predictions about the asymptotic behavior of large populations, as in [2, e.g.]. Results on the uniform convergence of empirical measures [18, e.g.] are employed to yield simpler proofs of min-max Simpson-Cramer majority [5,2] and yolk shrinkage [26]. The analysis suggests a rule of thumb as to whether or not a prediction based on distributional analysis will be valid for large finite populations. From the experimental point of view, the discussion helps clarify the mathematical underpinnings of statistical analysis of empirical voting data. A careful reading shows Tullock's original paper [28] to be consistent with the analysis given here.

20 Distribution/Availability of Abstract [X] unclassified/unlimited [ ] same as report [ ] DTIC users	21 Abstract Security Classification Unclassified
22a Name of Responsible Individual C. A. Tovey	22b Telephone *(Include Area Code)* (408) 646-2140 — 22c Office Symbol OR/Tv

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted — All other editions are obsolete

security classification of this page

Unclassified

A Critique of Distributional Analysis In Social Choice

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December 4, 1990
revised January 15, 1991
(to be issued as
Naval Postgraduate School Technical Report NPSOR-91-16)

[1]supported by a National Science Foundation Presidential Young Investigator Award ECS-8451032. A portion of this work was done while the author held a National Research Council Research Associateship at the Naval Postgraduate School

1 Introduction

Distributional analysis has been a widely used technique in the study of social choice in Euclidean models [28,29,1,3,8,15,5,2,23] (see also [4] and [19, Chaps. 11-12]) for more than two decades. In distributional analysis, a continuum or infinite population of voters is analyzed, where the population follows some probability distribution μ.

Infinite populations do not exist. Therefore, the principal purpose of distributional analysis must be to give insight into the behavior of large but finite populations.

In this paper it is shown that distributional analysis is flawed when applied to this end. The problem is essentially one of convergence: if the limiting case is to give insight into the large finite case, behavior of the latter should converge to behavior of the former as the population grows. Unfortunately, it turns out that properties of finite populations do not in general converge to the properties of infinite populations. In some cases a distributional analysis will predict that a point is in the core with probability 1, while the true probability converges to 0. Thus analysis of infinite populations may fail to yield any information about finite populations, however large.

An alternative technique

An alternative probabilistic technique for the study of social choice is termed here the *finite sample* method. In this method, n points are independently sampled from the distribution μ. This random finite sample from μ forms a *configuration* of n points whose properties are analyzed. A typical question would be: "what is the probability, as a function of n, that the configuration generated has nonempty core?" Typical answers to these questions are bounds or asymptotically close estimates for the desired probability.

It is sometimes possible to combine distributional analysis with finite sample analysis to make correct predictions about the asymptotic behavior of large populations. An example of this is found in [2]. We expose some key properties which enable the convergence in this case, enabling a simpler and more general proof of the convergence of min-max majority rule. We also estimate the population size for which the results are meaningful, *i.e.*, at which convergence begins to take hold. For committee sizes of 10,000 or more, a 2/3 majority rule is likely to be stable, under the concavity assumptions of [2]. For committee sizes of 250 or less, there is some doubt as to whether 2/3 majority rule is necessarily stable.

Following a suggestion due to Robert Foley, Richard McKelvey, and Gideon Weiss, we explore the use of uniform convergence theorems to transform distributional results into finite sample results. Theorems about the uniform convergence of empirical measures [18, e.g.] yield a simpler and more general proof of Simpson-Kramer min-max convergence[2] and a simpler though less general proof of yolk shrinkage[26]. The analysis suggests a rule of thumb as to when one might expect distributional analysis to give accurate or inaccurate predictions about the behavior of finite populations.

A careful reading of Tullock's original paper [28] reveals a clear insightful distinction between the distributional and finite sample methods, and a remarkable foreshadowing of some of the outcomes of finite sample analysis.

Empirical study of social choice

Another motivation for analyzing the distributional method, besides the clarification of results in the literature, is to help uncover a rigorous foundation for statistical empirical study of group choice. One would like to poll the members of a committee, assembly, or population (or in some other way extract data on their positions on the issues), and based on that data and some solution concept, make a prediction with some confidence regarding what the outcome will be. How do we experimentally test a solution concept? Ignoring the difficulties of data acquisition (e.g. sincerity), and any computational issues, there is still a problem regarding the stability of the solution concept with respect to individual perturbations. In other words, a person's views on issues are not perfectly constant; one can even change one's mind in the voting booth. How can we know that a prediction based on polls taken one day is apt to be close to the actual results the next day?

We may think of each person's views as having a probability distribution. When we interview a person we get a random sample from this distribution. When that person votes or negotiates in committee, it is on the basis of another random sample from this distribution. The problem is to establish rigorously the stability of a solution concept under these conditions.

In statistical terms, the finite sample from μ yields an empirical measure

μ_n. A solution concept is a statistic, a function f operating on probability measures. If f is a consistent statistic, then the limiting behavior of $f(\mu_n)$ will (almost surely) be like $f(\mu)$ and the solution concept is stable.

This issue has received a great deal of attention for the classical core or Nash equilibrium under the term "structural stability". The outcome is negative: the Nash solution concept is not usually applicable, and is never structurally stable in three or more dimensions [20]. In section 6 we illustrate how theorems for the uniform convergence of empirical measures [18, e.g.] can be invoked to establish the stability of other more widely applicable concepts.

The outline of the paper follows: the remainder of this section reviews essential definitions of the spatial model. Section 2 introduces the two methods by way of a small example. Section 3 analyzes the distributional method. Section 4 demonstrates in greater detail a case from [1] where the distributional method gives a misleading result. Section 5 discusses a case (the 64%-rule of Caplin and Nalebuf [2]) where the finite sample method may be combined with the distributional method to achieve results valid for large finite populations. Large is argued to be somewhere between 250 and 10,000 in this case. Section 6 introduces the use of uniform convergence of empirical measures and discusses in general when one may expect the distributional method to be useful and when we may expect it to be misleading. Section 7 concludes by re-examining Tullock's original paper [28].

1.1 Definition of the spatial model

In the Euclidean spatial model, a social choice involving m issues is to be made. The possible proposals are represented as vectors in $\Re^m$. Each individual i has a most preferred point $x_i \in \Re^m$. This point will be referred to as a voter point, or simply a voter. Under Euclidean preferences, an individual faced with two alternatives will select the one closest to her most preferred point, under the Euclidean norm. This model is more general than it appears: Davis *et al.* [3] show it is equivalent to any linearly transformed spatial model which maintains the properties of an inner product; Grandmont [8] (see also [2, section 5]) observes that the essential property of the Euclidean model is often the "division-by-hyperplane" property (in the Euclidean case, the perpendicular bisector of two points separates those who prefer one point to the other), and so results in the Euclidean model usually apply to the more general class of "intermediate preferences", including constant elasticity of substitution (C.E.S.) utility functions (these extend the class of Davis *et al.* by allowing a change to an L^p norm from the L^2 norm).

2 Two methods and an example

Let us begin with a simple two-dimensional model based on an example in [23]. Let μ be a probability distribution that is uniform on a circle (the circumference of a disk). Place a single voter v_1 at the center of the circle, which for convenience we locate at the origin. Randomly generate $n-1$ additional voter points $v_2, \ldots, v_n$, where n is even, according to μ.

We introduce some terminology. A particular realization of this random process is a *configuration*, a specific set of points $V = \{v_1, \ldots, v_n\}$ with specific locations in $\Re^m$. In this case V is a *finite configuration*. If $|V|$ is infinite V is an *infinite configuration*.

the finite sample method

Next we illustrate the method of finite sample analysis on the model just stated. The question we pose is: what is the probability that v_1 is undominated in the configuration V? A result of Schofield's [23] implies that the probability is positive, but the exact probability was not known until recently: v is undominated with probability $1/2^{n-2}$. Notice that the answer to the question is parameterized by n, as one would expect. The proof is sketched here since it will be needed in Sections 4 and 5.

Theorem 1 Place v_1 at the origin and generate $v_2, \ldots, v_n$ independently according to any nondegenerate sign-invariant distribution μ. Then for all even n, the probability v_1 is undominated is $1/2^{n-2}$.

Proof:([25]) Associate for each θ, $0 \leq \theta < \pi$, a line passing through the origin and an associated orientation. See Figure 1. Denote this line by $L(\theta)$. The open half space the line is oriented towards is the "front" and the other open half space is the "back" of the line $L(\theta)$.

Since the points are drawn from a nondegenerate distribution, the probability is 0 that any pair of the points $v_2, \ldots, v_n$ are collinear with the origin. Henceforth we assume this event does not occur.

For any $0 \leq \theta < \pi$ define the gap function $g(\theta)$ to equal the number of voter points in the front half plane of $L(\theta)$ minus the number of voter points in the back of $L(\theta)$. If $g(\theta) = -1$ or 1, the line $L(\theta)$ divides the $n-1$ points $v_2, \ldots, v_n$ as equally as possible given that n is even. If however the gap function $g(\theta)$ ever attains $|g(\theta)| \geq 3$ then one side of the line will contain at least $1 + n/2$ points and v_1 will not be a core point.

Starting at $\theta = 0$, increase θ continuously to π. Because no two points are collinear with v_1, $g(\theta)$ will change by either +2 or -2 as the line $L(\theta)$ crosses over a point v_i. Let $\theta_1, \ldots, \theta_{n-1}$ denote the values of θ at which $L(\theta)$ crosses over a voter and let $X_i = +2$ or -2 accordingly as the ith crossover increases or decreases $g(\theta)$. The key observation is that the gap function executes a random walk as θ goes from 0 to π.

Lemma 1. $X_1, \ldots, X_{n-1}$ are independent identically distributed variables taking values 2 with probability 1/2 and -2 with probability 1/2.

Proof of Lemma: the proof follows easily from the sign-invariance of μ. See Figure 2: the regions I and II are equally likely to contain the next point as we sweep $L(\theta)$ around. Details are given in [25].

There are 2^{n-1} possible paths for the random walk of the X_i to take. Of these, only two will keep the gap function at $|g(\theta) \leq 1|$. These are the alternating paths $X_i = 2(-1)^i$ and $X_i = -2(-1)^i$. Any nonalternating ~~sequence~~ sequence must contain two consecutive +2's or two consecutive -2's. If this ever happens, the gap function will change by 4 and so must leave the range $\{-1, 1\}$. By Lemma 1, each of the 2^{n-1} possible paths occurs with equal probability $1/2^{n-1}$. Therefore the probability that $\max_\theta |g(\theta)| \leq 1$ is $2/2^{n-1} = 1/2^{n-2}$ as desired. This completes the proof of Theorem 1.

A few remarks about Theorem 1: the proof only assumes μ is sign-invariant: $\mu(y) = \mu(-y)$, so it applies to the uniform rectangle distribution in [29,1] and many others. For the model under discussion, Theorem 1 gives a stronger outcome, for obviously (w.p.1) no other point in $\Re^2$ can be undominated. Thus the configuration V has nonempty core with exact probability $1/2^{n-2}$.

the distributional method

Now let us illustrate the distributional method on the same model. (The following closely follows analyses in [29,23,3,8]). Assume a continuum of voters uniformly distributed on the circle. Every line passing through 0 has mass $\leq 1/2$ on either side of it. That is, each halfspace h defined by a line through 0 has $\mu(h) \leq 1/2$. Thus v is undominated. In fact by [3, theorem 1] or [15, theorem 2] it is the unique "dominant" or undominated point. (The reader who is concerned about the "extra" point at 0 may observe that this only improves the position of 0 with respect to equilibrium.)

The contrast between the two methods is evident. The finite sample method shows that the probability of 0 being undominated, indeed of a nonempty core, rapidly converges to 0. The distributional method says that for an infinite population, the probability of 0 being undominated is 1.

The example of this section reveals that there is a flaw in the distributional method. It would be desirable for the outcome of the distributional method to coincide with the limiting behavior of finite samples, since the goal must be insight into the behavior of finite populations. Yet there could hardly be less consonance than in the example just given. In the next section we analyze the distributional method to explain how this flaw arises.

3 An analysis of the distributional method

A contrast with distributional analysis

We have observed that the outcomes of the two methods can differ. Let us point up an important distinction in how they operate. The distributional method works directly with μ, and quantities such as $\mu(h)$ are considered. On the other hand, in the finite sample method a configuration V is drawn from μ, and quantities such as $|V \cap h|$ are considered. Informally, the distributional method counts up voters by looking at the distribution function μ directly, while the finite sample method counts up voters by looking at configurations drawn from μ.

More formally, the distribution function μ analyzed in finite sample analysis is not an infinite configuration, rather it is a probability measure defined on the appropriate Ω, which for fixed n may be thought of as the set of all possible configurations of cardinality n. In contrast the distributional method treats μ as an infinite configuration.

A brief history of distributional analysis

~~In the literature~~, the term *distribution* is used in the economics literature to mean both "configuration" and "distribution function" as defined here. If we examine the literature of distributional analyses, we find that it is intertwined with analyses giving necessary and/or sufficient conditions for domination, local equilibrium, and/or global equilibrium in finite configurations (to use terminology defined here) [17,15,3,1,23,16]. For instance, Plott's classic paper [17] is titled

> A notion of equilibrium *and its possibility* [emphasis added] under majority rule.

Plott performs no probabilistic analysis but observes (quite rightly) [IBID, page 792] that

> it would only be an accident (and a highly improbable one) if an equilibrium exists at all.

Tullock's analysis [28] is, as Davis *et al.* [3, page 148] observe, "informally developed without theorems or proofs by the device of insightful examples." Later papers such as [15,3,16] meld these analyses by formalizing ideas of Tullock [28,29] and simultaneously generalizing Plott's results to infinite populations and/or more general preference functions (also global rather than local equilibrium). For instance, Davis *et al.*[3, page 148] contrast their work with Plott's since the latter

> allows only a finite number of individuals to be considered.

Presumably Davis *et al.* view this "limitation" of Plott's analysis as undesirable because more insight is needed as to the behavior of large finite populations.

In 1981 however Tullock remarks [30, page 190] that his analysis was "not regarded as very reliable any more because McKelvey proved that majority voting can reach any part of the issue space." The analysis Tullock refers to ultimately showed (see [22,23,20, e.g.]) that the set of configurations for which equilibrium exists is measure 0, for $d \geq 3$ and also for $d = 2$ and odd n, confirming what Plott had said all along. These powerful results seem implicitly to invalidate the distributional analyses. Yet, this consequence does not even now appear to be fully assimilated in the literature. The only unresolved case was $d = 2, n$ even, (and that was my original motive for undertaking this line of research.)

Where is the flaw in distributional analysis?

We have seen that some of the distributional analyses suggested implications at odds with the instability theorems of McKelvey, Schofield, Rubenstein, and others[12,13,21,20], So is there a flaw in the distributional arguments, and if so what is it? The crucial part is lucidly exposed by Arrow in his 1969 paper [1]. Summarizing Tullock's analysis, Arrow writes [page 108]:

> He [Tullock] assumes
> (1) that the number of voters is large, so large that we may consider them to constitute a continuum.

This assumption seems innocuous enough. In the mathematics literature, passing to the limiting continuous case is a popular technique. The problem is that majority rule requires us to evaluate $n/2$ where n = the number of voters, but the value $\infty/2$ is not well-defined. More precisely, if 0 is undominated and only one voter is located at 0 then placing two additional voters together at any location $x \neq 0$ must make 0 dominated (by the point ϵx for sufficiently small $\epsilon > 0$. But if n is treated as infinite no shifting of any finite number of voters changes the analysis, since $\infty/2 + 1 = \infty/2$.

What happens is that *a new definition is needed* when passing from the finite to the infinite case. Let us examine a specific definition from the literature. In an article by Davis, DeGroot, and Hinich [3], necessary and sufficient conditions are derived for the existence of a dominant point. As stated earlier, this analysis, unlike Plott's, is intended to apply to infinite populations. The critical definition of a non-dominance relation R is quoted below [3, page 149].

> Let P^* denote the distribution of most preferred points of the individuals. Let X be the most preferred point of an individual chosen at random from the population. [*note P^* is referred to as an infinite configuration in the previous sentence and as a probability function in the next sentence*] Given a (Borel) set $S \subset E_n$, Pr(S) will denote the probability that $X \in S$ under the distribution P^*.

Definition 1: For any points $y \in E_n$ and $z \in E_n$, it is said that yRz if $\Pr(||y - X|| \leq ||z - X||) \geq \frac{1}{2}$.

The definition of the relation R just given is mathematically unambiguous and therefore is mathematically correct. The mathematics in the paper [3] is of course correct. But there is a problem with the interpretation of the mathematical results. In [3] the passage just cited continues with the following interpretation:

> In other words, yRz if and only if at least half the population either prefers y to z or is indifferent between y and z.

What does the word "population" mean in the sentence just quoted? If we take it to mean the probability measure, then it would be accurate to say that

> yRz if and only the measure (mass) of the subset of the population, that either prefers y to z or is indifferent between y and z, is at least 1/2.

But if the word "population" refers to a finite sample drawn from the distribution P^*, then the meaning of yRz is given by the following theorem.

Theorem 2. Suppose a finite number of points are drawn at random according to the distribution P^*. Then

> yRz if and only if the probability is at least 1/2 that at least half the population either prefers y to z or is indifferent between y and z.

Proof: Suppose yRz. If we were to take a finite sample under the distribution P^*, each sample point would with probability at least 1/2 be at least as close to y as to z. Then the number of points in the sample at least as close to y as to z follows a binomial distribution with "success" parameter $p \geq 1/2$. From the most elementary properties of the binomial distribution $p \geq 1/2$ implies the probability is at least 1/2 that at least half the outcomes are "successes". Conversely, if the probability is at least 1/2 that at least half of the Bernoulli trials end in success, it must be that the parameter $p \geq 1/2$, whence yRz.

3.1 The heart of the problem

We have arrived at the heart of the problem. When going from finite to infinite populations, a new definition of the nondominance relation R was needed. Succinctly, let $\mathcal{A}$ denote "at least half the population either prefers y to z or is indifferent between y and z." Then for any finite sample population, *yRz means that $\mathcal{A}$ occurs with probability 1/2.* But the interpretation for infinite populations in [3] is, *yRz means that $\mathcal{A}$ occurs.*

If the purpose of the mathematical analysis of infinite populations in [3] is to gain insight into the behavior of large finite populations, then there should be a closer correspondence between the meanings of yRz for finite samples and for infinite populations.

The gap between the finite sample (Theorem 2) and the distributional (Definition 1) methods just discussed is between 1/2 and 1. In the earlier example of section 2 involving Theorem 1, the gap was (asymptotically) between 0 and 1. The larger gap in that example was due to the intersection of many events each with probability 1/2.

4 An unsuccessful case: The Sonnenschein-Arrow Theorem

Let us now examine a specific case of analysis from the literature where the predictions of distributional analysis are misleading. In his article, Arrow continues by stating a theorem (he attributes to Sonnenschein) that generalizes Tullock's example [1, pages 108–109]:

> For any pair of alternatives x, y, let $N(x, y)$ be the number of individuals who prefer x to y. Then let xMy be the statement $N(x, y) \geq N(y, x)$ and $x\bar{M}y$ the statement that $N(x, y) > N(y, x)$....
>
> **Theorem.** *Suppose that, for each alternative x^0, the set of alternatives x for which xMx^0 is closed, and [suppose] the set of alternatives [x] for which $x\bar{M}x^0$ is convex. Then for any compact (closed and bounded) convex set of alternatives S, there is (at least) one alternative x in S such that xMy for all y in S.*

Arrow later points out that "the hypotheses of the theorem are obviously fulfilled in Tullock's example." [IBID, page 110]. This is of course correct, *but only subject to assumption (1) above.* For if we employ the finite sample method of this paper, we find the *probability converges to 0 that the hypotheses of the Sonnenschein-Arrow theorem are fulfilled in Tullock's example.* The following theorem states and proves this statement precisely.

Theorem 3. Under the hypotheses of Tullock's example or of Theorem 1, the probability that the set $\left[x : x\bar{M}0\right]$ is convex converges to 0 as $n \to \infty$.

Proof: It suffices to consider only the more generous assumption of Theorem 1. Recall the gap function $g(\theta)$ used in the proof of Theorem 1. If (and only if) $g(\theta) > 1$ then a strict majority of the points are in the halfplane defined by the normal vector v_θ with orientation θ. Then for sufficiently small $\epsilon > 0$, the point $y = \epsilon v_\theta$ dominates 0, i.e. y is in the set $[x : x\bar{M}0]$. Rotate the vector v through an open halfplane, i.e. let θ range in the interval $[0, \pi)$. If the gap function $g(\theta)$ ever exceeds 1, drops to 1 (or below), and later exceeds 1 again, the set $[x : x\bar{M}0]$ will fail to be convex (see Figure 3). This is because there will exist distinct values $0 \leq \theta_1 < \theta_2 < \theta_3 < \pi$ such that for all sufficiently small $\epsilon > 0$, (ϵ, θ_1) and (ϵ, θ_3) are in the set, but (ϵ, θ_2) is not in the set (using (r, θ) notation). If the random walk executed by the gap function behaves in this fashion, then the set is not convex, and we call the walk "bad".

By Lemma 1, the values of the gap function execute an unbiased random walk centered around 0. Therefore we may select the orientation of $\theta = 0$ so that the walk has $n - 2$ steps and starts at 1. By the recurrence properties of one dimensional symmetric random walks [6, e.g.], the walk is bad with probability 1 as $n \to \infty$. In fact it will be bad infinitely often, so the set $[x : x\bar{M}0]$ will have many nonconvexities. This proves the theorem.

It has previously been observed that the Sonnenschein-Arrow Theorem can fail to be applicable. Greenberg [9], in a lovely paper on d-majority equilibrium, gives a deterministic example with $n = 4$ voters in which the set $[x : x\bar{M}0]$ is not convex. At the time it must have seemed that examples such Greenberg's would become less likely as n increased. For instance Kramer [11, page 313] remarks,

> Several authors, ... have argued that this instability is a "small-sample" problem, and that majority equilibria will be more

likely when the number of voters is large; examples and results supporting this thesis have been exhibited by ....

Theorem 3 demonstrates that Greenberg's example represents the rule, not the exception.

5 A successful case: The 64%-rule

Although the distributional method can mislead, it sometimes gives perfectly accurate predictions of the asymptotic behavior of finite populations. An excellent example is found in a recent paper by Caplin and Nalebuf [2]. They consider a class of voting procedures, parameterized by $0 \leq \delta \leq 1$, in which the status quo or incumbent can only be defeated or dislodged if more than δ of the population supports the contesting alternative. Caplin and Nalebuf first employ the distributional method: they show that if the distribution function μ is concave, then the smallest δ that guarantees an equilibrium (undefeatable) point, called the Simpson-Cramer min-max majority, is $1-(m/(m+1))^m$.([2, Theorem 2]). They continue and prove ([2, Theorem 3], essentially the same result is apparently found in [5, 2.4(iii),pp.151–152, 5.3 p.164]) that if a finite sample of size n is drawn at random from the concave distribution μ, then the min-max majority of the sample converges to the min-max majority of μ *a.e.*. Hence, "the bounds of the paper extend to large finite populations drawn from a concave density"[2, page 801]. Thus the distributional method is a success in this case.

One must take some care in applying the bounds to the finite case. Consider a uniform population density on an equilateral triangle (see Figure 4). The mass of μ in the shaded region is 5/9; it follows that the chances are close to 50% that more than 5/9 of a random sample will fall in the shaded region. But if this occurs, the center will not be a 5/9-majority core point, however slightly the sample fraction exceeds 5/9.

In fact a stronger statement is true: the triangle center is a 5/9-majority rule point with (asymptotic) probability no more than 1/8.

Theorem 4. Let n ideal points be generated independently from the uniform distribution on a regular triangle. Let p_n denote the probability that the triangle center is a 5/9 majority point. Then $\limsup_n \{p_n\} \leq 1/8$. Proof: see appendix.

Theorem 4 does not negate Theorem 2 of [2] in a substantial way. To begin with, there is the possibility that some other point very close to the triangle center is undefeated. But more importantly, suppose that for any $\epsilon > 0$, a $(5/9 + \epsilon)$-majority rule were employed. Then, by the almost sure convergence of Theorem 3 of [2], the probability converges to 1 that the triangle center is a majority point.

One of the beautiful things about Theorem 2 of [2] is the dimension-free corollary that $1 - 1/e$-majority rule will have a core, (which leads to the title of the paper). Since for any fixed dimension m, there exists an $\epsilon > 0$ such that $1 - (m/(m+1))^m + \epsilon \leq 1 - 1/e$, the analog of Theorem 4 is false for the dimension-free corollary. That is, an immediate and very nice consequence of Theorem 3 of [2] and its corollary is the following:

Corollary: Let n points be sampled independently from any concave distribution on $\Re^m$. Then the probability converges to 1, as $n \to \infty$, that the centroid of the distribution is a $1 - 1/e$-majority rule point.

How rapid is the convergence? In the case of a sign-invariant distribution in two dimensions, proposition 1 below states we can certainly expect an error of order $1/\sqrt{n}$.

Proposition 1. Under the conditions of Theorem 1, the largest majority that can be mustered against the origin has expected value $\geq \frac{(n+\sqrt{n})}{2}$.

Proof: From the proof of Theorem 1, the gap function executes a random walk around 0. The expected absolute distance from 0 at the end of a random walk is $\sqrt{n}/2$ [6]. Dividing by the population size n gives the result.

I have not been able to determine rigorous lower bounds in general. If the region is triangular instead of circular, the random walk is not stationary (in fact it is no longer Markovian), but heuristically we can again expect the maximum gap to be on the order of $\sqrt{n}$ in expected value from the largest distributional gap. The convergence theorems cited in the next section will tell us that the error levels can be expected not to exceed $O(1/\sqrt{n})$.

With a committee size of 100 (e.g. U.S. Senate), $1/\sqrt{n}$ is a fairly substantial 10%. If we seek an explanation for the stability of 2/3-majority rule in a group of this size, therefore, concavity is not quite enough. Concavity together with a limitation to 2 issues ($m = 2$ dimensions) might suffice. Alternatively, the extreme cases of triangular or simplicial distributions may in reality be quite rare.

If the population size is 10,000 or more, drawn from a concave density, the probability of stability under majority rule appears to be fairly good. From Proposition 1 we heuristically may expect that the maximum gap will usually not exceed several multiples of the expected value $\sqrt{n}/2$, say $6(\sqrt{n}/2) = 3\sqrt{n}$. At $n = 10,000$ this gap as a fraction of population is $3\sqrt{10000}/10000 \approx 3\%$. But however high the policy space dimension m, there is always a "cushion" of about 3% between $2/3$ and $1 - 1/e$. On the oher hand, When the population size is $n = 250$ or less, equilibrium may be unlikely. This is because Proposition 1 suggests that a gap of at least $\sqrt{n}/2$ will occur quite often. We then have $\sqrt{250}/2(250) \approx 3\%$, so the cushion is not big enough unless additional restrictions are placed on the preferences of the voter population.

Thus the min-max majority results of [2], particularly the dimension-free bounds, provide a successful application of distributional analysis to large finite populations, though some care must be taken in applying the results to smaller committee sizes.

6 General clues

Why do the distributional results discussed in section 5 apply to large finite populations, while those discussed previously do not? Part of the answer has to do with the difference between non-dominance and strict dominance. Recall from section 4 that the finite sample meaning of the non-dominance relation R does not converge to the meaning in the distributional case. In contrast, the strict dominance relation P : yPz iff yRz and not yRz does converge. That is, if yPz in the distributional sense, and a random sample of n points is taken, then yPz with respect to that finite sample with probability converging to 1 as $n \to \infty$. (This follows immediately from the weak law of large numbers and Davis *et al.*'s observation that "yPz if and only if $\Pr(||y - X|| < ||z - X||) > 1/2$.")

This difference is not enough. For example, suppose distribution μ is uniform in a square centered at y. Then for all $z \neq y$, yPz in the distributional sense. But if a finite sample of size $2n$ is taken, then by

Theorem 1 with probability converging to 1 there will exist $z \neq y$ such that zPy. However, suppose y strictly dominated all z in some compact set Z. We might then argue, if μ were continuous, that the strict domination occured with a minimum gap of some $\delta > 0$. If we could then find a way to reduce consideration Z to a finite, relatively small (e.g. polynomial in n) number of points, we could establish the desired behavior of the finite sample. These ideas are found in the proof of Theorem 3 in [2], where Compactness and Lemma 1 (page 807) provide~~s~~ the reduction to a finite number $(n+1)$ of points. Similar ideas are found in [26], where the fundamental basis extreme point theorem of linear programming provides the reduction to a finite number.

The preceding suggests that the mathematical tools for the convergence of empirical measures may be appropriate to these questions[1]. This turns out to be the case. The interested reader should consult chapter 2, "Uniform Convergence of Empirical Measures" of Pollard's excellent book[18]. A couple of the most pertinent results are cited below (specialized to our case and adapted to our terminology):

Definition. Let n points be drawn at random according to a probability measure μ on $\Re^m$. The *empirical measure* μ_n is that which places mass $1/n$ at each of the n points (obviously they need not be distinct.)

Let $\mathcal{C}$ denote a class of sets in $\Re^m$. For any $c \in \mathcal{C}$, it follows that $\mu_n(c)$ simply equals the fraction of the points which fell in c. The class $\mathcal{C}$ of most interest to us is the set of all closed and open halfspaces. Accordingly, let

$$\mathcal{C} = \left\{ c : c = [p \cdot x \leq p^0]; p \in \Re^m, p^0 \in \Re \right\}. \tag{1}$$

Also let $\mathcal{C}^+ = [\bar{\mathcal{C}}]$, the set of open halfspaces, and let $\mathcal{D} = \mathcal{C} \cup \mathcal{C}^+$. The uniform convergence theorem of [18] implies that the empirical measure converges to μ over these classes.

Theorem 5. Let μ be a probability measure on $\Re^m$. Then

$$\sup_{d \in \mathcal{D}} |\mu_n(d) - \mu(d)| \rightarrow 0 \text{ almost surely} \tag{2}$$

Proof: this follows from Theorem 14 (page 18), Lemma 15(i,ii)(page 18), and Lemma 18 (pages 20–21) of [18].

[1] I am indebted to Bob Foley, Richard McKelvey, and Gideon Weiss for suggesting this line of attack

This means that even if we consider all half-spaces h, the largest gap between the fraction of points falling in the half-space, and the expected fraction ($\mu(h)$), converges to 0.

To demonstrate the usefulness of Theorem 5, we invoke it to prove the convergence of the min-max majority. The first part of Theorem 6 generalizes Theorem 3 of [2] from bounded continous to arbitrary distributions, the second part of Theorem 6 is very similar to 2.4(iii) and Proposition 10 in [5]. Yet the proof of Theorem 6 is much shorter and simpler. This confirms the appropriateness of this line of attack (and the wisdom of my colleagues).

Theorem 6 Let μ be a probability measure on $\Re^m$. Let n points be randomly independently sampled from μ. Then the min-max majority value of the sample, $\alpha(\mu_n)$ converges to the distributional min-max majority $\alpha(\mu)$ almost surely. If in addition μ is continuous and possesses unique min-max winner point z, then the min-max winner of the sample converges a.s. to z.

Proof: If z is an α-majority point with respect to μ then by Theorem 5 it will be an $\alpha + \epsilon$-majority point for μ_n eventually, for any positive ϵ. Thus $\limsup\{\alpha(\mu_n)\} \leq \alpha(\mu)$. Conversely, for any $\beta < \alpha(\mu)$, set $\delta = \alpha(\mu) - \beta$. For all $x \in \Re^m$, there exists a hyperplane h_x through x such that a halfspace h_x^+ defined by h_x has mass $\mu(h_x^+) \leq \beta + \delta$. Again by Theorem 5, the supremum of the fractional discrepancies over all these halfspaces converges to 0 a.s. Thus,

$$\inf_x |\mu_n(h_x^+)| > \beta + \delta/2 \tag{3}$$

eventually, with probability 1 (a fraction of at least $\beta + \delta/2$ can be mustered against every point.) Hence $\liminf_n\{\alpha(\mu_n)\} \geq \alpha(\mu)$. This proves the first part of Theorem 6.

The proof of the first part has moreover established that z has limiting minimal winning supermajority fraction α. It remains to show that no points other than z can also be winning with fraction α. Accordingly let $\epsilon > 0$ be arbitrary. Let $S \subset \Re^m$ be an enormous ball containing z and with $\mu(S) > \alpha$, so that eventually with probability 1 no point outside S can be an α-majority winner. Let T denote S with the small ball of radius ϵ around z removed, $T = S \setminus B(z, \epsilon)$. By the compactness of T and continuity of μ, there exists β such that the minmax majority over all $x \in T$ equals

β. By the uniqueness of z, $\beta > \alpha$. Then by the same argument as led to inequality 3, eventually with probability 1 we have:

$$\inf_{x \in T} \mu(h_x^+) > \beta - \delta/2 > \alpha$$

Hence eventually no point in T will be an α-majority winner. This completes the proof.

Theorem 6 ensures convergence of $\alpha(\mu_n)$ holds for *any* distribution. This is of particular importance for empirical applications, because spatial voting data is often discrete. For example, the Senate data in [10] and other studies [24] are taken from roll call votes. Similarly, most public opinion polls ask yes/no questions or limit answers to integers in a small range (e.g. 1–5). In all these cases the real data will be discrete. Even if kernel smoothing ([18, pp. 35,42]) were employed the resulting distributions might not be continuous. Also notice the following: if *two* groups of samples were taken from μ, Theorem 5 would ensure the convergence of the two empirical measures to each other. This matches the scenario described in section 1, where information from polls or past voting records is used to predict an outcome.

In general, we consider a function(al) f whose domain is the set of probability measures and whose range is the reals. For example, f might be an indicator function for the event "0 is undominated", or f_i might be the ith coordinate of the center of mass of the distribution. When f is continuous, the uniform convergence of the empirical measure will ensure the convergence of $f(\mu_n)$ to $f(\mu)$.

Consider the indicator function just defined. It is not continuous, in the following sense: there exists $\epsilon > 0$ such that for all $\lambda > 0$, there exist empirical distributions μ_n and $\tilde{\mu}_n$ satifying

$$\sup_{d \in D} |\mu_n(d) - \tilde{\mu}_n(d)| \leq \lambda$$

but $|f(\mu_n) - f(\tilde{\mu}_n)| > \epsilon$. (Just take $\epsilon = .9$). Moreover the discontinuity occurs just at the distributions of interest, where the fraction on one side of a hyperplane is 1/2. From a more general point of view, this explains the failure of finite behavior to converge to distributional behavior as discussed in sections 3 and 4.

The mathematical guideline for convergence is the continuity of the functional. Let us attempt to formulate a less technical rule of thumb to give a general sense of how to make accurate predictions for finite populations based on distributional results: if the event or quantity of interest depends on the precise way voters are split among regions, then a convergence problem is apt to arise; if it relies instead on having a certain fraction or more in a region, then the result is apt to apply to the large finite case, possibly with the fraction perturbed slightly.

Let us apply these observations to the yolk radius convergence shown in [26]. A hyperplane is median if the two closed halfspaces it defines each contains at least half the population. The yolk is the smallest ball intersecting all median hyperplanes [7,14]. If there is a simple majority rule core point the yolk is that point. Under what circumstances can we expect the yolk radius to be small? From a distributional point of view[2], a yolk radius of 0 corresponds to a nonempty core. Necessary and sufficient conditions for a nonempty core, in the distributional sense, are (see [3,15]) that μ be *weakly centered*: every hyperplane through 0 is a median hyperplane. Therefore a distributional analysis predicts that weak centeredness would be necessary and sufficient for the yolk radius of random samples to converge to 0.

Our rule of thumb suggests that there may be a problem with the exact 50:50 split of the weak centeredness condition, but that a $(50+\epsilon):(50-\epsilon)$ splitting condition would be apt to work. It turns out that the true necessary and sufficient condition is that μ be *strictly centered*[26]: for every hyperplane *not* passing through 0, the halfspace it defines not containing the origin must contain strictly less than half the population. This outcome seems well in accord with the guidelines proposed above.

We can invoke Theorem 5 to prove the sufficiency half of this result[3], though under an additional assumption of continuity of the distribution μ. Despite the lessened generality of Theorem 7, the ease and brevity of its proof are noteworthy. **Theorem 7.** Let n points be sampled independently from μ, a strictly centered continuous distribution on $\Re^m$. Then the radius of the yolk of the sample converges to 0 *a.s.* as $n \to \infty$.
Proof: see Appendix.

[2] this distributional analysis is due to Richard McKelvey

[3] the essentials of this proof were suggested to me independently by Robert Foley, Richard McKelvey, Loren Platzman, and Gideon Weiss.

7 Rereading Tullock's paper on the general irrelevance...

The results in this paper might seem to invalidate claims in Tullock's original work. A careful reading shows this is not so. Tullock's original paper, "The general irrelevance of the general impossibility theorem" [28], is in my opinion an altogether brilliant piece of work, combining important empirical evidence (the scarcity of actual cycling or chaos) with abundant creative inspiration and exceptional mathematical intuition (as well as dramatic exposition). A careful reading reveals that Tullock is actually discussing finite configurations, and only appeals to the infinite configurations as an intuitive aid. For example, after describing a uniform distributional model, Tullock writes ([28, page 259]):

> This might be called the perfect geometrical model, in which the number of voters whose optima fall in a given area is exactly proportional to its area. Given that the voters are finite in number, small discontinuities would appear. Two areas that differ little in size might have the same number of voters; indeed, the smaller might even have more. Cycles are, therefore, possible, but they would become less and less important as the number of choosing individuals increases.

Later, Tullock specifically remarks that the probability of cycling should increase as the population grows [IBID, page 261]:

> For close to the center, the area which is preferred to A would be farther from the center than A. Cycling becomes more probable. When we get very close to the center a point randomly selected from among those which could get a majority over the given point would have a good chance of being farther from the center than it is. At this point, however, most voters will feel that new proposals are splitting hairs, and the motion to adjourn will carry.

This intuitive statement is in accord with Theorem 1. Thus Tullock is not claiming that cycles won't usually exist in large populations [4]. Tullock's main point is that they *won't matter*.

One of the arguments Tullock advances to support his point is that unless proposals were carefully manipulated, "the voting process would in all probability lead to rapid movement toward the center [28, page261]. This argument is actually a loose forerunner of the yolk, the smallest ball intersecting all median hyperplanes. (Tullock's discussion of intersections of median lines, pages 261–262, is especially evocative of the yolk.)

Since that time the yolk has been rigorously established by Ferejohn, McKelvey, and Packel [7] and McKelvey [14]. More recently it has been proved that the radius of the yolk does converge to 0 a.s. for the distribution of Tullock's example (or any other centered distribution) [26]. Considering the length of time by which Tullock's work preceded the mathematical development of the appropriate technical tools, Tullock's insights seem all the more remarkable.

8 Acknowledgments

The author thanks George Dantzig, Bob Foley, John Gilmour, Richard McKelvey, Loren Platzman, Norman Schofield, Richard Stone, Gideon Weiss, Lyn Whitaker, and Kevin Wood, for invaluable discussions and correspondance.

9 Appendix: Proof of Theorems 4 and 7

Proof of Theorem 4: The three lines through the triangle center in Figure 5 divide the triangle into the six regions labelled a, b, c, d, e, f. For notational ease, let the region label also represent the number of sample points falling in that region. If the center is to be a 5/9-majority point, then $b + c + d \leq 5/9$, and similarly $d + e + f \leq 5/9; f + a + b \leq 5/9$. These imply our key inequalities: $b - e \leq 1/9; c - f \leq 1/9; d - a \leq 1/9$. That is,

[4] He also argues that "it is possible, by simple majority voting, to reach points at almost any portion of the issue space", an adumbration of the classic chaos theorems of McKelvey and Schofield [12,13,21,22]

the number of points in each rhombus is no more than $n/9$ more than the number of points in the opposing triangle. Applying the strong law of large numbers, the actual number in each region, for large n, will be within $O(\sqrt{n})$ of its expected value with very high probability (geometrically decreasing chance of failure). We may therefore condition on the partitioning among the three rhombus-triangle pairs being close to the expected value of $n/3$ in each of these three paired regions, and the error in our resulting estimate converges to 0. Once we condition on this likely event, the three key inequalities become independent. Now approximating the binomial distribution of parameters $\sim n/3, 1/3$ with a normal distribution, (by the strong law of large numbers), and since $n^{1/2}$ dominates $n^{1/4}$, it follows that the probability is asymptotically 1/2 that the gap between rhombus and opposing triangle of the three inequalities. (In other words the median and mean of the binomial are very close). Therefore, the conditional probability that that three key inequalities all hold is asymptotically 1/8. Thus p_n in the limit is bounded by 1/8. This proves Theorem 4.

The upper bound of 1/8 in Theorem 4 can be extended easily to $1/2^{m+1}$ for m dimensions.

I would moreover conjecture that $p_n \to 0$ as $n \to \infty$.

Theorem 7[5] Let n points be sampled independently from μ, a strictly centered continuous distribution on $\Re^m$. Then the radius of the yolk of the sample converges to 0 *a.s.* as $n \to \infty$.

Proof: Following the proof in [26], we show that the largest distance from 0 to any median hyperplane converges to 0. Since this distance is an upper bound on the yolk radius, the result will follow.

For any $x \neq 0$, let h_x^+ denote the halfspace not containing the origin defined by the hyperplane normal at x. By strict centeredness $\mu(h_x^+) < 1/2$. By continuity $\mu(h_x^+)$ is continuous in x.

Let $\epsilon > 0$ be arbitrary. Clearly the largest vote attained against 0 by points ϵ or more away from 0 is attained by points ϵ away, or more accurately

$$\sup_{||x|| \geq \epsilon} \mu(h_x^+) = \sup_{||x|| = \epsilon} \mu(h_x^+).$$

By compactness of the set the latter supremum is taken over, and continuity,

[5]see the acknowledgment footnote, page 18.

the supremum is attained. Thus there exists $\beta < 1/2$ such that for all $||x|| \geq \epsilon$, we have $\mu(h_x^+) \leq \beta$.

The halfspaces h_x^+ are contained in the class $\mathcal{C}$. Let the n points be sampled from μ. Apply Theorem 5 to find that with probability 1, as n increases,

$$\mu_n(h_x^+) \leq \frac{\beta + 1/2}{2} < 1/2 \forall ||x|| \geq \epsilon.$$

This implies that there is no median hyperplane at distance ϵ or more from 0, whence the result follows.

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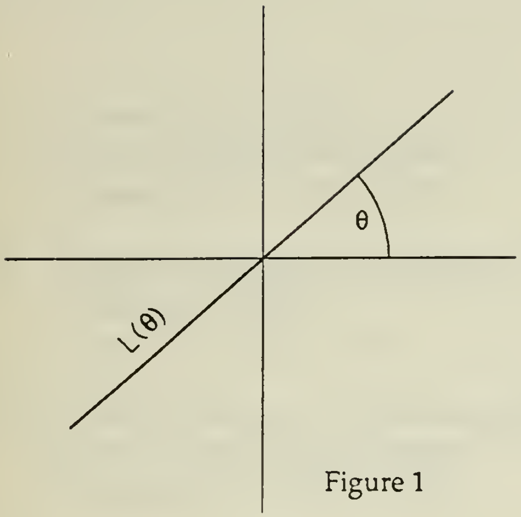

Figure 1

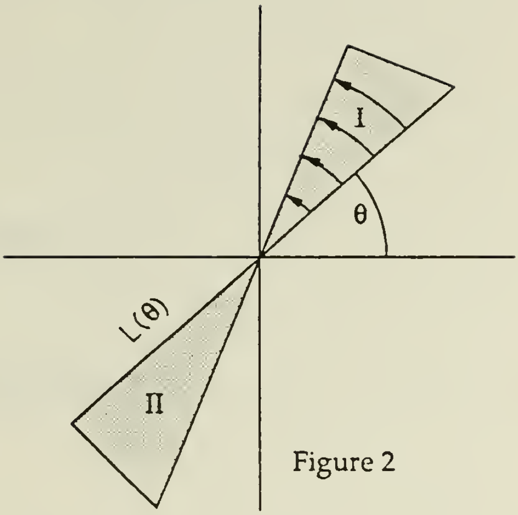

Figure 2

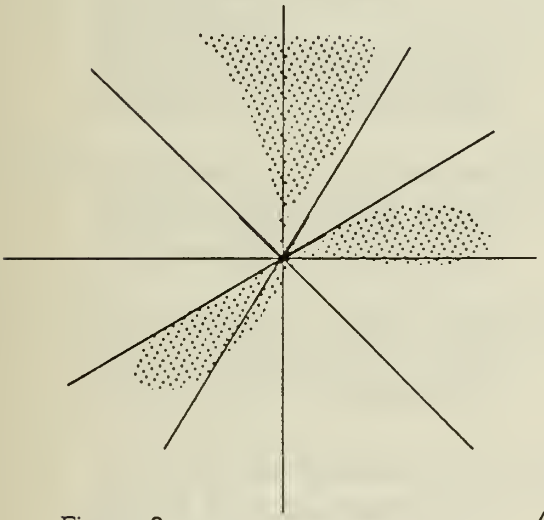

Figure 3

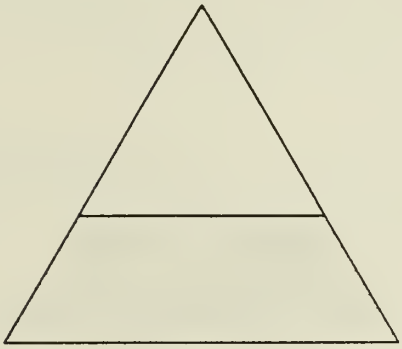

Figure 4

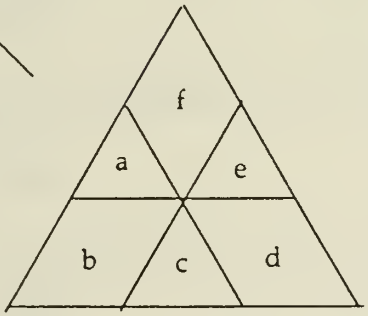

Figure 5

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